Comparison Between Cross-spectrum and Spectrum Average Generalized to Q-devices

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***Abstract*—This paper generalizes to q-devices the confidence interval comparison of the cross-spectrum and the spectrum average estimators. The probability density function of the cross-spectrum leads to the variance-gamma distribution for the common case of 2-devices. However it has no analytical solution from 3-devices, a solution is given using the Fourier transform of the product of characteristic functions.**

***Index Terms*—cross-spectrum; spectrum average; QR decomposition; characteristic function; Bayesian statistics; inverse problem; confidence interval.**

I. Introduction

This paper generalize to q-devices the comparison of the efficiency of the cross-spectrum(c-s) estimator, i.e. the average of pairs of 2 spectra covariance, in regards to the average(s.a) of these spectra. We assume that each spectrum is composed of a common red noise $R(f)$, called the signal, and a white noise $N(t)$. This red noise can be originated from gravitational waves on the line of sight from pulsars observations. It then affects the phase of time of arrival of the pulses measured by q independent radio-telescopes(RTs). We could also use a set of q clocks and interval counters observing the same clock and compute its periodogram, or even a set of q (or $q+1$) clocks to intercompare them. Each measurement instrument adds an intrinsic white noise which are assumed to be uncorrelated. Thus we want to assess the variance σ_R^2 of the sought signal in a bandwidth, i.e. the power in one bin of the power spectral density(PSD) as respects to the c-s and s.a estimators. The variance of the noise $\sigma_{N,i}^2$, defined as for the signal but for the i-th RT, requires a perfect knowledge to determine the probability density function(pdf) of both estimators. Furthermore the s.a estimator exhibit a sufficient statistic which means of minimal variance, and then could be more efficient. We propose a Bayesian study to define the 95% upper limit of the parameter σ_R^2 generalized for q RTs.

II. Statement of the estimators

Let us consider at a given Fourier frequency the spectra of the i-th RT as

$$X_i = R + N_i \tag{1}$$

where R is a normal complex rv of variance σ_R^2 i.e. the signal level and N_i are uncorrelated normal complex rv of variance $\sigma_{N,i}^2$ i.e. the white noise level of each RT. There is no reason to expect an intrinsec similar noise level for each device. Therefore the average weighted by the variances $\sigma_{N,i}^2$ for each device is the optimal estimator. Then the s.a and c-s estimators are respectively

$$\begin{aligned}\widehat{S_{\text{sa}}} &= \mathcal{R}\left[\sigma_\mu^2 \frac{\sum_i^q X_i}{\sigma_{N,i}^2}\right]^2 + \mathcal{I}\left[\sigma_\mu^2 \frac{\sum_i^q X_i}{\sigma_{N,i}^2}\right]^2 \\ \widehat{S_{\text{cs}}} &= \langle X_i \cdot \tilde{X}_j \rangle_m \qquad \text{with} \quad i \neq j\end{aligned} \tag{2}$$

where $\langle \cdot \rangle$ stands for the m average over the different combinations of RTs with $m = \binom{q}{2}$ and $\tilde{}$ stands for the complex conjugate of the quantity which is below. Moreover $\mathcal{R}[\cdot], \mathcal{I}[\cdot]$ stands respectively for the real and the imaginary parts, appearing through the Fourier transform operation, of the quantities within the brackets whereas q is the number of RTs. The factor σ_μ^2 is the noise variance weighting normalization corresponding to

$$\sigma_\mu^2 = \left(\sum_i^q \frac{1}{\sigma_{N,i}^2} \right)^{-1}. \tag{3}$$

Denoting $\mathbb{E}[\cdot]$ the mathematical expectation of the quantity within the brackets,

$$\begin{aligned}\mathbb{E}\left[\widehat{S_{\text{sa}}}\right] &= \sigma_R^2 + \sigma_\mu^2 \\ \mathbb{E}\left[\widehat{S_{\text{cs}}}\right] &= \sigma_R^2\end{aligned} \tag{4}$$

which means that the spectrum average estimator is **biased** [1].

III. Probability Density Function

A. Spectrum Average

The s.a estimator leads to the following χ^2 distribution with 2 degrees of freedom resulting from the real and imaginary part of the spectrum,

$$p(\widehat{S_{\text{sa}}}|\sigma_R^2) = \frac{e^{-\frac{\widehat{S_{\text{sa}}}}{2\sigma^2}}}{2\sigma^2} \tag{5}$$

where,

$$\sigma^2 = \tfrac{1}{2}(\sigma_\mu^2 + \sigma_R^2). \tag{6}$$

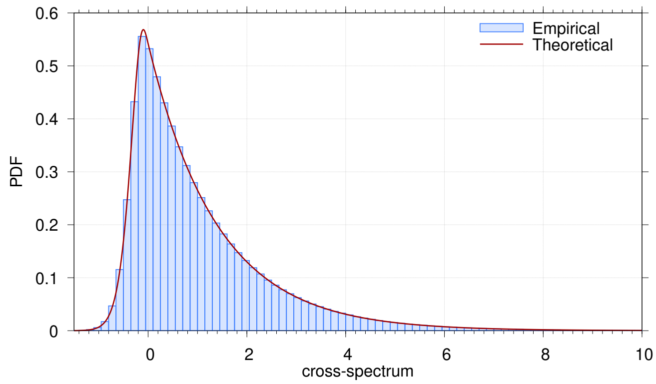

Fig. 1: Comparison of the empirical and theoretical PDF of the c-s for 5 RTs where the variances are $\sigma_R^2 = 1$ a.u. and $\sigma_N^2 = 2$ a.u.

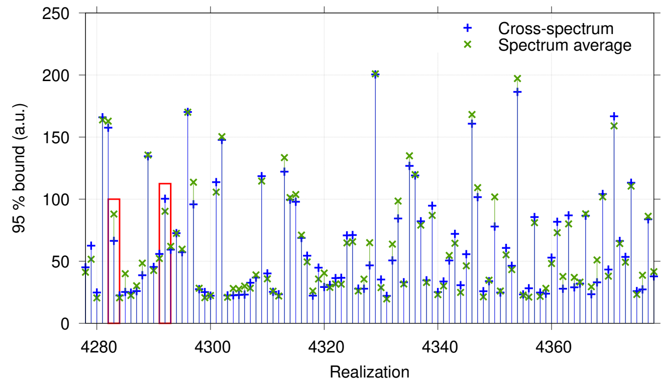

Fig. 2: Set of 100 realizations for 5 RTs of 95% bounds for cross-spectrum (blue +) and spectrum average (green ×) where $\sigma_R^2 = 6$ a.u. and $\sigma_N^2 = 10$ a.u.

B. Cross-spectrum

The c-s estimator leads to the variance-gamma (VΓ) distribution for 2 RTs as established in [2] but for more than 2 RTs it is no longer the case. Having no exact solution known nowadays, we give an approximation of it. First we perform a QR decomposition by using the Householder transformation in an orthogonalization process. Second we compute the eigenvalues λ_j of the resulting components and obtain a linear combination of χ^2 distribution as followed,

$$\widehat{S_{\text{cs}}} = \sum_{j}^{q} \lambda_j \chi_k^2 \tag{7}$$

where k is the number of degree of freedom of each eigenvalue. It can be shown that the white noises induce negative eignvalues where the signal implies a positive one. Then we define the characteristic function of χ_k^2 as

$$\phi_j(t) = (1 - 2i\lambda_j t)^{-k/2} \tag{8}$$

where i is the imaginary unit complex number and we apply a variable change of $-t$ for the negative eignvalues. The χ^2 distribution being independent, the characteristic function of the c-s becomes $\phi(t) = \prod_j^q \phi_j(t)$. The probability density function of the c-s is finally define as

$$p(\widehat{S_{\text{cs}}}|\sigma_R^2) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp^{-it\widehat{S_{\text{cs}}}} \phi(t)dt. \tag{9}$$

Figure 1 shows that theoretical probability density function fits very well the histogram obtained by 10^7 Monte Carlo simulations for 5 RTs. The variance of each white noise is the same $\sigma_N^2 = 2$ a.u. whereas the signal level is $\sigma_S^2 = 1$ a.u.

IV. Comparison of the 95% upper limit

Now we tackle the inverse problem from the direct problem, i.e. the statistics of the signal level knowing the s.a or c-s estimate. The Bayes theorem enables us to establish this link and the resolution will be detailed in the full paper. It is important to notice that each set of measurement for a given set of parameters leads to a different pdf. So we give an example of a hundred simulations as shown in Fig. 2 for 5 devices with $\sigma_R^2 = 6$ a.u. and $\sigma_N^2 = 10$ a.u. These results show that the s.a and c-s 95% upper bound are really close. The two red boxes shown on Fig. 2 highlights the fact that sometimes the c-s upper bound is lower than the s.a and sometimes it is the opposite with a significant difference. However on 10 000 simulations average the s.a gives a slightly more stringent upper bound and reaches the lowest value.

V. Conclusion

We proposed a method to compute the probability density function of the cross-spectrum involving q measurement devices or radio-telescopes based on the numerical integration of Fourier transform of the characteristic function product. This method has been proven by massiv Monte Carlo simulations. This permits to compare the efficiency of the spectrum average and the cross-spectrum estimator to assess the signal level. Even if the spectrum average is biased both estimators require the perfect knowledge of the noise level.

The spectrum average estimator being a sufficient estimator suggested that the spectrum avecrage estimator would be the more stringent and it is confirmed as it reaches the minimal upper limit. However it happens that the cross-spectrum gives a more string 95% upper bound with a non negligible difference. Thus it is wiser to compute both estimators and choose the lower upper limit.

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References

[1] E. Rubiola and F. Vernotte. The cross-spectrum experimental method. arXiv:1003.0113v1, March 2010.

[2] A. Baudiquez, E. Lantz, E. Rubiola, and F. Vernotte. Cross-spectrum measurement statistics: uncertainties and detection limit. *IEEE Trans. Ultras. Ferroelec. Freq. Contr.*, pages 1–1, june 2020.